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Isotopy–Isomorphy Loops of Prime Order*

ROBERT L. WILSON, JR.†

Department of Mathematics, University of Wisconsin, Madison, Wisconsin 53706

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1. Introduction and Definitions

(1.1) The main result in this paper is Theorem (2.4) which states that a loop of prime order is isomorphic to all of its loop isotopes if and only if it is a group. The proof is combinatorial and based on a formula due to Bryant and Schneider [2].

(1.2) We say a loop $(L, \cdot)$ has the *isotopy–isomorphy* property and write $(L, \cdot) \in I\text{–}I$ if $(L, \cdot)$ is isomorphic to all of its loop isotopes.

We use the following notation from [2].

(1.3) If $(L, \cdot)$ is a loop and $a, b \in L$ we denote by $L(a, b, \circ)$ the principal isotope of $(L, \cdot)$ whose operation $\circ$ is defined by

$$x \circ y = (x/b) \cdot (a\backslash y).$$

Any principal loop isotope of $(L, \cdot)$ (and hence up to isomorphism any loop isotope of $(L, \cdot)$) can be obtained as $L(a, b, \circ)$ for some choice of $a, b \in L$.

(1.4) We write $L_1 \overset{\theta}{\cong} L_2$ if the loop L_1 is isomorphic to L_2 under the mapping θ. For a fixed loop $(L, \cdot)$, let G be the set

$$\{\theta \mid L(a, b) \overset{\theta}{\cong} L(c, d) \text{ for some } a, b, c, d \in L\}.$$

Bryant and Schneider [2] show that G with composition forms a group.

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(1.5) Again for a fixed loop $(L, \cdot)$, let $A(a, b)$ denote the group of automorphisms of $L(a, b, \circ)$, and $A = A(1, 1)$. Note that A is a subgroup of G and hence $|A|$ divides $|G|$. For any loop $(H, *)$, let $N_\mu(H, *)$ denote the middle nucleus [1, p. 57] of $(H, *)$.

2. Counting Arguments

Theorem 5 of [2] shows that the number of principal loop isotopes of a finite loop $(L, \cdot)$ isomorphic to a particular principal loop isotope $L(a, b, \circ)$ is

$$\frac{|G|\,|N_\mu[L(a, b, \circ)]|}{|A(a, b)|}. \tag{2.1}$$

Using the fact [1, p. 57] that the middle nuclei of isotopic loops are isomorphic and hence have the same order, it is immediate that the numerator of (2.1) is actually independent of a and b, i.e.,

Proposition 2.2. *The number of principal loop isotopes of* $(L, \cdot)$ *isomorphic to* $L(a, b, \circ)$ *is*

$$\frac{|G|\,|N_\mu(L, \cdot)|}{|A(a, b)|}.$$

Theorem 2.3. *If* $(L, \cdot)$ *is an I–I loop of order* n *and* n^2 *does not divide* $|G|/|A|$, *then* $N_\mu(L, \cdot)$ *is not trivial.*

Proof. If $(L, \cdot)$ is an I–I loop, all n^2 of its principal loop isotopes are isomorphic to $L(1, 1, \cdot) = (L, \cdot)$. Thus by (2.2),

$$n^2 = \frac{|G|}{|A|}\,|N_\mu(L, \cdot)|.$$

By hypothesis, n^2 does not divide $|G|/|A|$ and hence $|N_\mu(L, \cdot)| > 1$.

Theorem 2.4. *A loop of prime order is an I–I loop if and only if it is a cyclic group.*

Proof. It is known [1, p. 57] that every group is an I–I loop. Thus we need only show that if $(L, \cdot)$ is an I–I loop of prime order p then $(L, \cdot)$ is associative. G is a subgroup of the symmetric group on L, and hence $|G|$ divides $p!$. Thus $|G|/|A|$ divides $p!$ Since p is prime, p *but not* p^2 divides $p!$. Hence p^2 does not divide $|G|/|A|$. By Theorem 2.3, $N_u(L, \cdot)$ is not trivial. For any loop the order of the middle nucleus divides the order of the loop, which in this case is prime, so $N_\mu(L, \cdot) = L$ and L is associative.

3. Nonprime Orders

In light of Theorem 2.4 one may ask whether the I–I property and associativity are equivalent for any nonprime orders. A construction by the author [3] shows that there is a nonassociative I–I loop for any even order for which there exist nonassociative loops, i.e., even orders > 4.

References

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